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ACTUATOR AND SENSOR PLACEMENT FOR STRUCTURAL TESTING AND CONTROL

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In this paper, the actuator and sensor locations of a structural test item are selected as a replacement of the disturbance inputs and the performance outputs of a real structure. The most straightforward approach is to place sensors in the areas of performance evaluation, and actuators in the areas of disturbance action. However, this solution is rarely possible due to technical and economical reasons. Therefore, the actuators and sensors need to be placed in preselected regions, and should duplicate as close as possible the disturbance action and the performance measurements. In this paper a placement problem with non-collocated actuators and disturbances, as well as non-collocated performance and sensor outputs, is solved. The solution is determined by locating sensors (actuators) such that the Hankel singular value vector of a structure from actuator inputs to sensor outputs is closely correlated with the Hankel singular value vector of the structure from the disturbance inputs to performance outputs. It is shown that this approach improves additionally the cross-coupling between actuators and performance, and between disturbances and the sensors, thus improving overall closed loop performance. The method is illustrated with the determination of sensors of a truss structure, where two selected sensors replaced an original set of 36 sensors.

1. INTRODUCTION

In this paper, we address the actuator and sensor placement problem as applied to flexible structures. For flexible structures the question of actuator and sensor location is of specific importance; their testing or control often requires the implementation of a large number of sensors and actuators that make the selection too complex or too expensive. It was shown in reference [1] that for flexible structures the placement algorithm is comparatively simple. However, it maximizes the controllability and observability of the modes under current test, rather than those actually excited in the real structure. Consider, for example, an antenna structure. The antenna as described in reference [2] is subjected to wind disturbances that act predominantly at the antenna dish. Antenna pointing accuracy is an ultimate measure of its performance. In structural tests it is difficult and expensive to apply dynamic forces at the dish to simulate the disturbances, and to measure the pointing accuracy. Thus, the test should be conducted such that the actuators located at the allowable sections of the structure will imitate the wind disturbances as closely as possible, and sensors selected from the candidate locations will detect the antenna motion that is the most relevant to the pointing accuracy.

A typical actuator and sensor location problem in structural testing can be described as follows. The test is planned using available information on the structure itself, on disturbances acting on the structure and on the expected structural performance. The first information is typically in the form of a structural finite element model. The disturbance

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information includes disturbance location and disturbance spectra. The structure performance is commonly evaluated through the displacements or accelerations at certain structural locations. The configuration of the plant in structural testing is represented in a block diagram as in Figure 1. In this diagram the structure input is composed of two inputs that are not necessarily collocated: the disturbance input (w) and the actuator input (u). Similarly, the plant output is divided into two sets: the performance output (z) and the sensor output (y). The actuators represent the inputs applied during a test. The disturbance inputs include disturbances, noises (known and unknown) and commands; generally they are not applied during the test. The sensor signals include structure outputs recorded during the test. The performance output contains signals that characterize the system performance, and is not necessarily measured during the test. In general, it is not possible to duplicate the dynamics environment during testing; not only due to physical restrictions or limited knowledge of disturbances, but also because the actuators cannot be located at the disturbance locations, and because sensors cannot be placed at the locations of performance evaluation. Thus, to obtain the performance of the test item close to the performance of a structure in the real environment, one uses the available (or candidate) locations of actuators and sensors and formulates the selection criteria and the selection mechanisms to imitate the real environment.

The control design problem of a structure can be defined in a manner similar to the structural testing. Namely, actuators are placed within the allowable locations, and they are not necessarily collocated with the disturbance locations; sensors are placed at the sensor allowable locations, generally outside the locations of performance evaluation. In the control nomenclature, u is the control input, y is the plant output accessible to the controller, w is the disturbance input and z is the vector of the performance output, see, for example, Boyd and Barrat [3].

The "classical" actuator and sensor problem statement considers the actuated input, and/or the sensed output only; see, for example, DeLorenzo [4], Gawronski [5], Gawronski and Lim [1], Lim [6], Lim and Gawronski [7], Maghami and Joshi [8], Salama *et al*. [9] and Skelton and DeLorenzo [10]. The disturbance inputs and performance outputs are either ignored or assumed collocated with the sensor and actuator locations, respectively. Lim [11] first approached this problem of the non-collocated inputs and outputs by introducing special weights to the actuator/sensor placement index that reflects the importance of disturbances.

In this paper, we address the two-input two-output actuator and sensor location problem as applied to flexible structures. We derive the placement rules based on the properties of the structural Hankel singular values, and illustrate their application with the truss sensor location.

2. MODAL REPRESENTATION

A flexible structure with n_d degrees of freedom is a linear system represented by the second order matrix differential equations

$$M\ddot{q} + D\dot{q} + Kq = B_o u, \qquad y = C_{oq}\, q + C_{ov}\, \dot{q}. \tag{1}$$

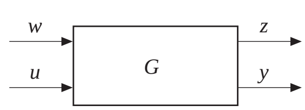

Figure 1. The structure configuration for testing and control.

In the equations, q is the $n_d \times 1$ displacement vector, u is the $s \times 1$ input vector, y is the output vector, $p \times 1$, M is the mass matrix, $n_d \times n_d$, D is the proportional damping matrix, $n_d \times n_d$, (see reference [12] for the definition of the proportional damping), K is the stiffness matrix, $n_d \times n_d$, the input matrix B is $n_d \times s$, the output displacement matrix C_{oq} is $p \times n_d$, and the output velocity matrix C_{ov} is $p \times n_d$. The number p is the number of outputs, and s is the number of inputs. The mass matrix is positive definite, and the stiffness and damping matrices are positive semidefinite.

Let $\Phi\ (n_d \times n)$ be the modal matrix consisting of n natural modes, $n \leqslant n_d$, $\Phi = [\phi_1\ \phi_2 \ldots . \phi_n]$. Introduce a new variable q_m, such that $q = \Phi q_m$. Substituting q from the above equation into equation (1), one obtains the modal mode

$$\ddot{q}_m + 2Z\Omega\dot{q}_m + \Omega^2 q_m = B_m u, \qquad y = C_{mq}\, q_m + C_{mv}\, \dot{q}_m \tag{2}$$

where $Z = M_m^{-1/2} K_m^{-1/2} D_m$, and $\Omega = M_m^{-1} K_m$ is a diagonal matrix of natural frequencies $\Omega = \mathrm{diag}\,(\omega_1, \omega_2, \ldots, \omega_n)$, and $M_m = \Phi^T M \Phi$, $K_m = \Phi^T K \Phi$ and $D_m = \Phi^T D \Phi$ are the diagonal modal mass, stiffness and damping matrices. The modal input matrix is $B_m = M_m^{-1}\, \Phi^T B_o$, while C_{mq}, and C_{mv} are the modal displacement and rate matrices, respectively; $C_{mq} = C_{oq}\, \Phi$ and $C_{mv} = C_{ov}\, \Phi$.

Introduce a state vector x, which consists of n modal components, $x^T = \{x_1^T\, x_2^T \ldots x_n^T\}$. The ith state component, x_i, is defined as

$$x_i = \begin{Bmatrix} \omega_i\, q_{mi} \\ \dot{q}_{mi} \end{Bmatrix}, \tag{3}$$

where q_{mi} and $\dot{q}_{mi}$ are ith modal displacement and velocity.

The triple (A, B, C) corresponding to the state vector x is the modal state space representation of a flexible structure. It has block-diagonal matrix A, and the related blocks of B and C:

$$A = \mathrm{diag}\,(A_i), \qquad B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix} C = [C_1 \quad C_2 \quad \ldots \quad C_n], \tag{4}$$

where $i = 1, 2, \ldots, n$, and where A_i, B_i and C_i are 2×2, $2 \times r$ and $s \times 2$ blocks, respectively:

$$A_i = \begin{bmatrix} 0 & \omega_i \\ -\omega_i & -2\zeta_i\, \omega_i \end{bmatrix}, \qquad B_i = \begin{bmatrix} 0 \\ b_i \end{bmatrix}, \qquad C_i = \begin{bmatrix} \dfrac{c_{qi}}{\omega_i} & c_{vi} \end{bmatrix}. \tag{5}$$

The triple (A_i, B_i, C_i) is the modal state space representation of the ith component. The ith modal component equation is

$$\dot{x}_i = A_i\, x_i + B_i\, u, \qquad y_i = C_i\, x_i, \qquad i = 1, \ldots, n. \tag{6}$$

The controllability and observability grammians are positive definite matrices W_c and W_o, defined as

$$W_c = \int_0^\infty \exp\,(At) BB^T \exp\,(A^T t)\, \mathrm{d}t, \qquad W_o = \int_0^\infty \exp\,(A^T t) C^T C \exp\,(At)\, \mathrm{d}t. \tag{7}$$

The system is balanced if its grammians are equal and diagonal; see reference [13]:

$$W_c = W_o = \Gamma, \qquad \Gamma = \mathrm{diag}\,(\gamma_1, \ldots, \gamma_N), \qquad \gamma_i \geqslant 0, \qquad i = 1, \ldots, N, \tag{8}$$

where the positive variable γ_i is the ith Hankel singular value of the system, and $N = 2n$ is the number of states.

The controllability and observability grammians of the modal representation are diagonally dominant, see reference [5]:

$$W_c \simeq \mathrm{diag}\,(w_{ci}\, I_2), \qquad W_o \cong \mathrm{diag}\,(w_{oi}\, I_2), \tag{9}$$

where w_{ci} and w_{oi} are given by

$$w_{ci} = \| B_i \|_2^2 / 4\zeta_i \omega_i, \qquad w_{oi} = \| C_i \|_2^2 / 4\zeta_i \omega_i. \tag{10}$$

The positive scalars $\| B_i \|_2$ and $\| C_i \|_2$ are defined as the input and the output gains of the ith mode, see reference [14], and $\| X \|_2$ is the Euclidean norm of X. The ith Hankel singular value γ_i is approximately a geometric mean of the ith grammians entries, $\gamma_i \cong \sqrt{w_{ci}\, w_{oi}}$, i.e.,

$$\gamma_i \simeq \frac{\| B_i \|_2 \, \| C_i \|_2}{4\zeta_i\, \omega_i}. \tag{11}$$

3. ACTUATOR–SENSOR PROPERTIES

Let (A, B, C) be the modal representation of a structure, with s inputs, p outputs, n components and $N = 2n$ states. For simplicity of presentation, the problem of sensor placement only is considered (actuator placement is similar). Denote by $\gamma_i\,(k)$ the Hankel singular value for the kth mode at the ith sensor location, and by $\gamma(k)$ the Hankel singular value for the kth mode at all sensor locations. The following is known from references [1] and [5].

Property 1:

$$\gamma^2(k) \cong \sum_{i=1}^{p} \gamma_i^2\,(k), \qquad k = 1, \ldots, n. \tag{12}$$

For the plant as in Figure 1, with inputs w and u, and outputs z and y, let G_{wz} be the transfer matrix from w to z, let G_{wy} be the transfer matrix from w to y, let G_{uz} be the transfer matrix from u to z, and let G_{uy} the transfer matrix from u to y. Let $\gamma_{wz}\,(k)$, $\gamma_{uy}\,(k)$, $\gamma_{wy}\,(k)$ and $\gamma_{uz}\,(k)$ be the kth Hankel singular values for the transfer functions G_{wz}, G_{uy}, G_{wy} and G_{uz}, respectively. The following multiplicative property holds.

Property 2:

$$\gamma_{wz}\,(k)\gamma_{uy}\,(k) \simeq \gamma_{wy}\,(k)\gamma_{uz}\,(k), \qquad k = 1, \ldots, n. \tag{13}$$

Proof. Denote by B_w and B_u the modal input matrices for w and u, respectively, and by C_z and C_y the modal output matrices for z and y, respectively; the structure is in the modal representation. The controllability and observability grammians for the kth mode are as follows, see equation (10):

$$w_{cw}\,(k) \cong \frac{\| B_{wk} \|_2^2}{4\zeta_k\, \omega_k}, \qquad w_{cu}\,(k) \cong \frac{\| B_{uk} \|_2^2}{4\zeta_k\, \omega_k}, \qquad w_{oz}\,(k) \cong \frac{\| C_{zk} \|_2^2}{4\zeta_k\, \omega_k}, \qquad w_{oy}\,(k) \cong \frac{\| C_{yk} \|_2^2}{4\zeta_k\, \omega_k}, \tag{14}$$

where B_{wk} and B_{uk} are the kth block-rows of B_w and B_u, respectively, and C_{zk} and C_{yk} are the kth block-columns of C_z and C_y, respectively. Note that

$$\gamma_{wz}^2(k) \simeq w_{cw}(k) w_{oz}(k), \qquad \gamma_{uy}^2(k) \simeq w_{cu}(k) w_{oy}(k), \tag{15a}$$

$$\gamma_{wz}^2(k) \simeq w_{cw}(k) w_{oy}(k), \qquad \gamma_{uz}^2(k) \simeq w_{cu}(k) w_{oz}(k), \tag{15b}$$

and introduce equation (15) to equation (13) to check that Property 2 holds. □

Property 2 shows that for each mode the product of Hankel singular values of the performance loop (from disturbance to performance) and the control loop (from actuators to sensors) is approximately equal to the product of the cross-couplings: from disturbances to sensors, and from actuators to performance output. The meaning of this property lies in the fact that by increasing the actuator–sensor connection, $\gamma_{uy}(k)$, one increases at the same time the cross-connection: for actuators-to-performance, and for disturbance-to-sensors. It also shows that sensors respond not only to the actuator input, but also to the disturbances, and actuators impact not only the sensors, but also the performance.

This property is useful in the closed loop design. For the plant as in Figure 1 we have

$$z = G_{wz} w + G_{uz} u, \qquad y = G_{wy} w + G_{uy} u. \tag{16}$$

The closed-loop transfer matrix G_{cl} from w to z, with the controller K such that $u = Ky$, is as follows:

$$G_{cl} = G_{wz} + G_{uz} K (I - G_{uy})^{-1} G_{wy}. \tag{17}$$

From the second part of the right-hand-side of the above equation, it follows that the controller impacts the closed loop performance not only through the action from u to y, but also through the cross-actions from u to z, and from w to y. Therefore, if the transfer matrices G_{wy}, and G_{uz}, are zero, the controller has no impact whatsoever on the performance z. Thus the controller design task consists of simultaneous gain improvement between u and y, w and y, and u and z. However, Property 2 shows that the improvement in the controllability and observability of G_{uy} automatically leads to the improvement of the controllability and observability of G_{wy}, and G_{uz}. Thus, the task of actuator and sensor location simplifies to the manipulation of G_{uy} alone.

4. IN A SEARCH FOR ACTUATOR AND SENSOR LOCATIONS

The above properties are the basis of the actuator and sensor search procedure. Denote by γ_{uy}^2 the vector of squares of Hankel singular values for all available sensors, and by γ_{wz}^2 the vector of squares of Hankel singular values for the disturbance input and the performance output. Then, a non-negative correlation coefficient ρ, between the γ_{uy}^2 and γ_{wz}^2 Hankel singular values, is defined as

$$\rho^2 = \frac{\gamma_{wz}^{2T} \gamma_{uy}^2}{\| \gamma_{wz}^2 \|_2 \, \| \gamma_{uy}^2 \|_2}. \tag{18}$$

It will serve as the actuator and sensor location performance index. Indeed, if $\gamma_{uy}^2 = \gamma_{wz}^2$, the sensor locations perfectly reproduce the disturbance-performance transfer function, and the index ρ achieves its maximal value $\rho = 1$; in this case the input–output controllability and observability properties are perfectly aligned (within a constant multiplier) with the controllability and observability properties of the disturbance-performance transfer function. Moreover, according to Property 2, the visibility of disturbances at the output improves, as well as the influence of the input on the performance is maximized. Similar to the total performance index, ρ, define the performance index ρ_i of the ith sensor:

$$\rho_i^2 = \frac{\gamma_{wz}^{2T}\gamma_{uyi}^2}{\|\gamma_{wz}^2\|_2\,\|\gamma_{uy}^2\|_2}, \qquad i = 1, \ldots, p, \tag{19}$$

where γ_{uyi}^2 is the vector of squares of Hankel singular values for the ith sensor. Note that $0 \leqslant \rho_i \leqslant \rho \leqslant 1$; it is non-negative because both γ_{wz}^2 and γ_{uyi}^2 are non-negative; it does not exceed 1, because γ_{uyi}^2 is a part of γ_{uy}^2. The question arises as to how the individual sensor coefficients ρ_i participate in the total one, ρ. The following property explains this question.

Property 3:

$$\rho^2 \cong \sum_{i=1}^{p} \rho_i^2. \tag{20}$$

Proof. From Property 1, one obtains that the square of the Hankel singular values of all sensor location is a sum of the squares of Hankel singular values for each individual sensor,

$$\gamma_{uy}^2 \cong \sum_{i=1}^{p} \gamma_{uyi}^2. \tag{21}$$

Introducing equation (21) to equation (18), and using notation (19) one obtains equation (20). □

This property shows that the index ρ for the set of sensors/actuators is an r.m.s. sum of indexes of each individual sensor or actuator. This decomposition allows for the evaluation of an individual sensor/actuator and its impact on the whole set performance.

For placement of a large number of sensors the maximization of the above performance index alone may not be a satisfactory criterion. Suppose that a specific location of a sensor gives high performance index ρ_i. Inevitably, locations close to it will also have high performance indices. However they are not necessarily the best choice, since the sensors at these locations can be replaced by the appropriate adjustment of gain of the original sensor. In this case one wants to find sensor locations that cannot be compensated by the original sensor. These locations can be determined using the additional correlation coefficient r_{ik}, defined as

$$r_{ik}^2 = \frac{\gamma_{uyi}^{2T}\gamma_{uyk}^2}{\|\gamma_{uyi}^2\|_2\,\|\gamma_{uyk}^2\|_2}, \qquad i = 1, \ldots, p, \qquad k = i+1, \ldots, p, \tag{22}$$

Denote a small positive number by ε, and define the membership index $I(k)$ for the kth sensor as

$$I(k) = \begin{cases} 0, & \text{if } r_{ik} > 1 - \varepsilon \text{ and } \rho_k \leqslant \rho_i, \quad \text{for } k > i, \\ 1, & \text{elsewhere.} \end{cases} \tag{23}$$

The index determines the acceptance of the kth sensor. If $I(k) = 1$, the kth sensor is accepted as the one that is not correlated with other sensors. If $I(k) = 0$ the kth sensor is rejected (in this case two locations i and k are either highly correlated, or the ith location has higher performance ρ_i). Based on the extensive simulation results it is recommended to use the values of ε from the range $\varepsilon = 0.01$–0.05.

4.1. PROCEDURE

Based on the derived properties, the following sensor search procedure is adapted.

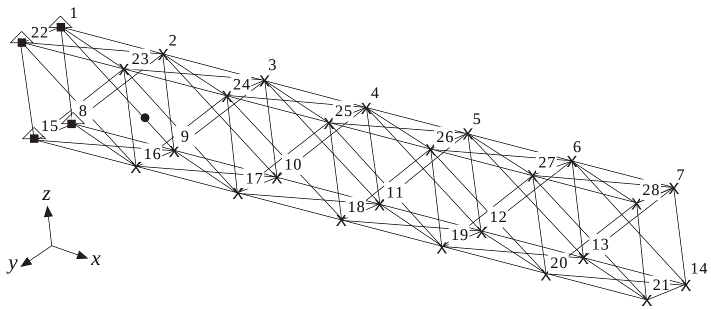

Figure 2. The truss.

(1) The Hankel singular value vectors are determined for the following transfer functions: G_{wz}, the transfer function from the disturbance to the performance, G_{uyi}, the transfer functions from all actuators to each individual sensor, $i = 1, \ldots, n$; and G_{uy}, the transfer function from all actuators to all sensors.
(2) The performance index ρ_i of each sensor is determined from Equation (19).
(3) The correlation coefficient r_{ik} from Equation (22), and the membership index $I(k)$, from Equation (23) are determined to check if the current location is highly correlated with the previously selected locations.
(4) If $I(k) = 1$ the sensor is selected, otherwise it is rejected.

5. EXAMPLE

Consider a steel truss as in Figure 2. For this truss, $l_1 = 10$ cm, $l_2 = 8$ cm, and the cross-sectional area is 1 cm^2. The disturbance w is applied at node 7 in y direction, the performance z is measured as rates of all nodes; the input u is applied at node 26 in the z direction, and the candidate sensor locations are at the nodes 5, 6, 7, 12, 13, 14, 19, 20, 21, 26, 27 and 28 in the x, y and z directions (a total of 36 locations). The task is to select a minimal number of sensors that would measure as close as possible the disturbance-to-performance dynamics.

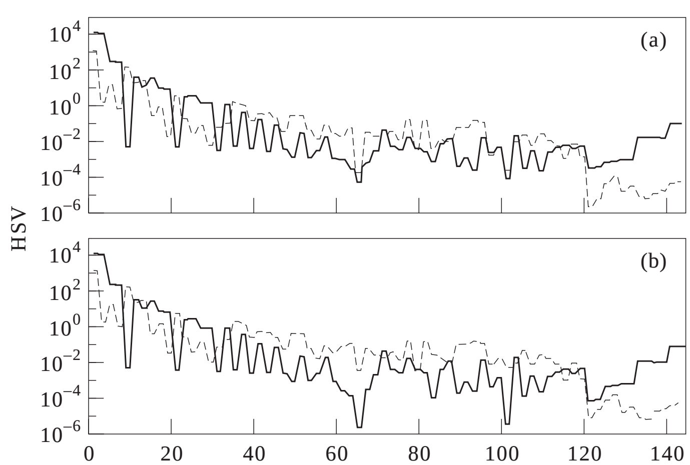

Figure 3. Hankel singular values of: (a) G_{wz} (—) and G_{uy} (----); (b) G_{wy} (—) and G_{uz} (----).

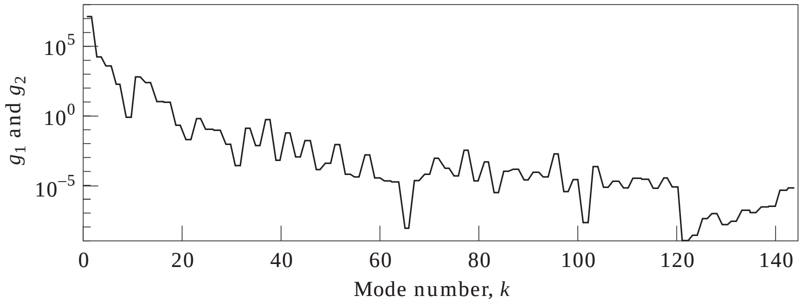

Figure 4. Overlapped plots of $g_1 = \gamma_{wz}\,\gamma_{uy}$ (—) and $g_2 = \gamma_{wy}\,\gamma_{uz}$ (----).

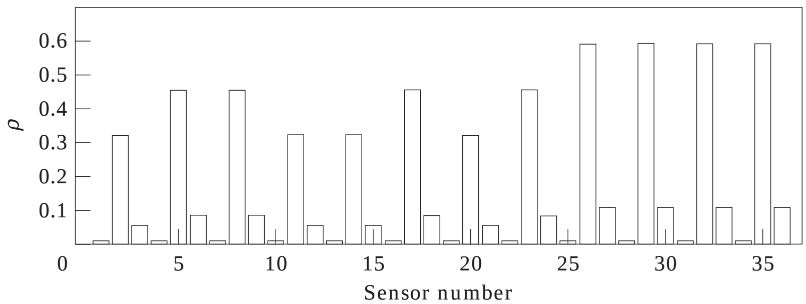

Figure 5. The correlation coefficient for each sensor.

First, the Hankel singular values of G_{wz}, G_{wy}, G_{uz}, and G_{uy} are determined, and presented in Figure 3. Next, Property 2 is checked. Equation (13) holds since the curves $g_1(k) = \gamma_{wz}(k)\gamma_{uy}(k)$ and $g_2(k) = \gamma_{wy}(k)\gamma_{uz}(k)$ overlap in Figure 4.

In the following, the correlation coefficients ρ_i for each sensor are determined from Equation (19), and their plot is shown in Figure 5. Note that although there are sensors with high values of ρ_1, they can be highly correlated. Therefore the membership index $I(k)$ is determined, assuming that $\varepsilon = 0{\cdot}03$. This index is shown in Figure 6. Its only non-zero values are for $k = 29$ and $k = 30$, corresponding to node 14, and directions y and z. Thus the rate sensors at node 14 in the y and z directions are chosen for this particular task.

6. CONCLUSIONS

Typically, the actuator and sensor placement problems are formulated such that the disturbances are collocated with the input, and performance is collocated with the output,

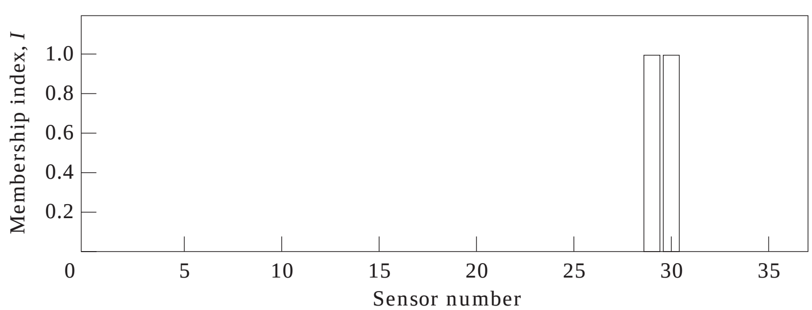

Figure 6. The membership index for each sensor.

or the actuator and sensor are placed such that the controllability and observability properties of all, or selected, modes are maximized. In this paper, a placement problem with non-collocated actuators and disturbances, as well as non-collocated performance and sensor outputs, is solved. The solution is determined by locating sensors (actuators) such that the Hankel singular values of the structure from actuator inputs to sensor outputs are as close as possible to the Hankel singular values of the structure from the disturbance inputs to performance outputs. It is shown that this approach also improves the cross-coupling between actuators and performance, and between disturbances and the sensors, thus improving overall closed loop performance. The search procedure involves the maximization of the correlation of the Hankel singular values vector of each sensor (actuator) with the Hankel singular values vector from the disturbance to the performance. The approach is illustrated with the determination of sensors of a truss structure, where two selected sensors replaced a set of 36 sensors.

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